ON POINTWISE-COMPACT SETS OF MEASURABLE FUNCTIONS

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The result proved below concerns a convex set of functions, measurable with respect to a fixed measure, and compact in the topology of pointwise convergence. The first and most interesting theorems along these lines were proved in [6] and [7] by A. Ionescu Tulcea. Several alternate proofs have been given since that time—for example [8]. The case of nonconvex sets was studied by Fremlin [4] and by Talagrand [10].

For the result proved here, I weaken the "separation property", and correspondingly weaken the conclusion, using the weak topology $\sigma(L^1, L^\infty)$ rather than the metric topology of L^1 or L^0. The result is then applicable to the proof of the recent characterization of Pettis integrability in terms of the "core".

The following notation will be fixed throughout the paper. Let $(\Omega, \mathfrak{F}, \mu)$ be a complete probability space. $\mathcal{L}^0 = \mathcal{L}^0(\Omega, \mathfrak{F}, \mu)$ denotes the set of all real-valued measurable functions. $L^0 = L^0(\Omega, \mathfrak{F}, \mu)$ denotes the space of equivalence classes obtained by identifying functions that agree almost everywhere. Similar distinctions apply to $\mathcal{L}^1$, L^1, $\mathcal{L}^\infty$, L^∞. The topology on $\mathcal{L}^0$[or L^0] is induced by the pseudometric [or metric] defined by

$$d(f, g) = \int |f - g| \wedge 1 \, d\mu \quad .$$

If A is a subset of Ω, the topology (on $\mathbb{R}^\Omega$) of pointwise convergence on A will be denoted $\tau_p(A)$. Thus a net f_α of functions converges to f according to $\tau_p(A)$ iff $f_\alpha(a) \to f(a)$ for all $a \in A$. If W is a subset of $\mathcal{L}^0$, we write $(W, \mathcal{L}^0)$ for the topological space with point set W and topology obtained from the pseudometric on $\mathcal{L}^0$. Similarly, if $W \subseteq \mathcal{L}^1$, we write $(W, \mathcal{L}^1)$ and $(W, \sigma(\mathcal{L}^1, \mathcal{L}^\infty))$ for W equipped with the strong and weak topologies (respectively) of $\mathcal{L}^1$.

The following hypotheses will be in effect through most of this paper: Let W be a subset of $\mathcal{L}^0$. Let $E \subseteq \Omega$. Suppose the following <u>separation property</u> holds: If $f, g \in W$, then $f = g$ on E if and only if $f = g$ a.e.

To reduce confusion, I will also use these two notations. Let $W_1 = \{f|_E :$

$f \in W\} \subseteq \mathbb{R}^E$, and let W_2 be the image of W under the quotient map $\mathcal{L}^0 \to L^0$..
The separation property says that the identity map $W \to W$ induces a bijection $W_1 \to W_2$.

The first proposition is essentially due to Ionescu Tulcea. The proof is carefully spelled out here to show exactly the sort of reasoning that is involved.

PROPOSITION 1. Suppose W and E are as above. If W is $\tau_p(\Omega)$-countably compact, then W_2 is closed in L^0 and the evaluations $f \mapsto f(e)$ are $\mathcal{L}^0$-continuous on W for $e \in E$. That is, the identity map $(W, \mathcal{L}^0) \to (W, \tau_p(E))$ is continuous.

Proof. Let $f_n \in W$, and assume $f_n \to f$ $(\mathcal{L}^0)$. There is a subsequence (f_n') with $f_n' \to f$ (a.e.). But W is $\tau_p(\Omega)$-countably compact, so (f_n') has a cluster point $g \in W$ for the topology $\tau_p(\Omega)$. Thus $f = g$ a.e. This shows W_2 is closed in L^0.

Now fix $e \in E$. Suppose $f_n, f \in W$ and $f_n \to f$ $(\mathcal{L}^0)$. I claim that $f_n(e) \to f(e)$. Suppose not. Then there is a subsequence (f_n') of (f_n) so that $f_n'(e)$ converges, but not to $f(e)$. Then there is a subsequence (f_n'') of (f_n') such that $f_n'' \to f$ (a.e.). Let $g \in W$ be a $\tau_p(\Omega)$-cluster point of (f_n''). Then $g(e) = \lim f_n'(e) \neq f(e)$, while $g = \lim f_n'' = f$ a.e., contradicting the separation property. This shows $f \mapsto f(e)$ is $\mathcal{L}^0$-continuous on W. □

Note. Suppose the measure space $(\Omega, \mathfrak{F}, \mu)$ has this property: if (f_n) is a sequence in $\mathcal{L}^0$, and every subsequence has a measurable $\tau_p(\Omega)$-cluster point, then there is a subsequence that converges a.e. In that case, in the above proposition, the identity map $(W, \mathcal{L}^0) \to (W, \tau_p(E))$ is a homeomorphism. Fremlin [4] has shown that all perfect measure spaces have this property.

In the next theorem, the case $E = \Omega$ was proved by Ionescu Tulcea [6].

PROPOSITION 2. Suppose W and E are as above. If W is $\tau_p(\Omega)$-sequentially compact, then the natural map $(W_1, \tau_p(E)) \to (W_2, L^0)$ is a homeomorphism. So the identity map $(W, \tau_p(\Omega)) \to (W, \mathcal{L}^0)$ is continuous.

Proof. First, I claim that W_2 is compact in L^0. Let $f_n \in W$, and suppose $f_n \to h$ $(\mathcal{L}^0)$. There is a subsequence (f_n') of (f_n) with $f_n' \to h$ (a.e.). There is a subsequence (f_n'') of (f_n') and $g \in W$ with $f_n'' \to g$ $(\tau_p(\Omega))$. Then $h = g$ a.e. Thus (W_2, L^0) is compact.

Next, since $(W, \tau_p(\Omega))$ is sequentially compact, it is countably compact, so by Proposition 1, the natural map $(W_2, L^0) \to (W_1, \tau_p(E))$ is continuous. But (W_2, L^0) is compact and $(W_1, \tau_p(E))$ is Hausdorff, this natural map is a homeomorphism. □

A set $W \subseteq \mathcal{L}^0$ is uniformly integrable iff for every $\epsilon > 0$, there exists $a < \infty$ so that

$$\int_{\{|f| > a\}} |f| \, d\mu < \epsilon$$

for all $f \in W$. In particular, W is bounded in the $\mathcal{L}^1$ norm.

Here is the main result of the paper. Its proof is not difficult.

PROPOSITION 3. Suppose W and E are as above. If W is convex, uniformly integrable, and $\tau_p(\Omega)$ - countably compact, then the two topologies $\tau_p(E)$ and $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$ coincide on W. So the identity map $(W, \tau_p(\Omega)) \to (W, \sigma(\mathcal{L}^1, \mathcal{L}^\infty))$ is continuous.

Proof. Let $e \in E$ and $r \in \mathbb{R}$. The (image in W_1 of the) set $\{f \in W: f(e) \leq r\}$ is closed in $(W_1, \tau_p(E))$, and hence, by Proposition 1, closed in (W_2, L^1). It is therefore closed in (L^1, L^1). But it is convex, so it is closed in $(L^1, \sigma(L^1, L^\infty))$, and therefore closed in $(W_2, \sigma(L^1, L^\infty))$. Similar assertions can be made for a set $\{f \in W: f(e) \geq r\}$. Thus the map $f \mapsto f(e)$ is $\sigma(L^1, L^\infty)$-continuous on W_2. Thus the natural map $(W_2, \sigma(L^1, L^\infty)) \to (W_1, \tau_p(E))$ is continuous. Now W is uniformly integrable and W_2 is closed in L^1, so $(W_2, \sigma(L^1, L^\infty))$ is compact [1, IV.8.11]. So the map $(W_2, \sigma(L^1, L^\infty)) \to (W_1, \tau_p(E))$ is a homeomorphism, and thus the identity map $(W, \sigma(\mathcal{L}^1, \mathcal{L}^\infty)) \to (W, \tau_p(E))$ is a homeomorphism. □

Notes. (a) It follows in particular that $(W_1, \tau_p(E))$ is sequentially compact.

(b) Under these hypotheses it does not follow in general that the topologies $\mathcal{L}^1$ and $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$ coincide on W. A counterexample of Talagrand [10] is also a counterexample to this.

(c) The stronger conclusion that the topologies $\mathcal{L}^1$ and $\tau_p(E)$ coincide on W is true if the measure space $(\Omega, \mathfrak{F}, \mu)$ has this property: if (f_n) is a sequence in $\mathcal{L}^0$, and every $\tau_p(\Omega)$ - cluster point of (f_n) vanishes a.e., then $f_n \to 0$ in measure. Fremlin's theorem [4] shows that perfect measure spaces have this property.

The proofs of the following two corollaries are left to the reader. Corollary 4 is essentially due to Ionescu Tulcea [7].

COROLLARY 4. Suppose W and E are as above. Suppose that $E = \Omega$ and that W is convex and $\tau_p(\Omega)$ - countably compact. Then the two topologies $\tau_p(\Omega)$ and $\mathcal{L}^0$ coincide on W. If, in addition, W is uniformly integrable, then the three topologies $\tau_p(\Omega)$, $\mathcal{L}^1$, $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$ coincide on W.

The "separation hypothesis" on W and E is not postulated in the next Corollary.

COROLLARY 5. Let W be a uniformly integrable, convex, $\tau_p(\Omega)$-compact subset of $\mathcal{L}^1$. Define

$$A = \bigcap \{\omega \in \Omega : f(\omega) = g(\omega)\}$$

where the intersection is over all pairs $f, g \in W$ with $f = g$ a.e. Assume that $A \cap \{\omega : f(\omega) \neq g(\omega)\} \neq \emptyset$ if $f, g \in W$ and $\mu\{\omega : f(\omega) \neq g(\omega)\} > 0$. Then the identity map $(W, \tau_p(\Omega)) \to (W, \sigma(\mathcal{L}^1, \mathcal{L}^\infty))$ is continuous. In particular, for any $B \in \mathfrak{F}$, the map $f \mapsto \int_B f \, d\mu$ is $\tau_p(\Omega)$-continuous on W.

The following Corollary is due to Tortrat [11].

COROLLARY 6. Let X be a Banach space, $\mathfrak{F}$ the Baire sets of (X, weak), and μ a probability measure on $\mathfrak{F}$. If μ is τ-smooth, then there is a separable subspace A of X with μ-outer measure 1.

Proof. Let W be the unit ball of the dual space X^*. Define $A = \bigcap \{f^{-1}(0) : f \in W, f = 0 \text{ a.e.}\}$. Then by τ-smoothness, A has outer measure 1. By Corollary 4, with $E = \Omega = A$, the topologies $\tau_p(A)$ and $\mathcal{L}^0$ coincide on W. Thus $(W, \tau_p(A))$ is metrizable, so the weak* topology on the dual ball of A is metrizable, so the subspace A is separable. □

Note. From this can be deduced well-known theorems of Gothendieck and Phillips; see [2, Theorem 5.1].

The following is a result of Talagrand; partial results were proved by Geitz [5] and by Sentilles [9].

PROPOSITION 7. Let X be a Banach space, and $\varphi : \Omega \to X$ scalarly measurable. Assume $\{f \circ \varphi : f \in X^*, \|f\| \leq 1\}$ is uniformly integrable. Suppose $\text{cor}_\varphi(C) \neq \emptyset$ for all $C \in \mathfrak{F}$ with $\mu(C) > 0$, where

$$\text{cor}_\varphi(C) = \bigcap \{\text{cl conv } \varphi(C \setminus N) : N \in \mathfrak{F}, \mu(N) = 0\} .$$

Then φ is Pettis integrable.

Proof. Consider a measure space $(\Omega', \mathfrak{F}', \mu')$ defined by: $\Omega' = X$, $\mathfrak{F} = \text{Baire}(X, \text{weak})$, $\mu' = \varphi(\mu)$. Let W be the unit ball of X^*; this is a convex, uniformly integrable, subset of $\mathcal{L}^0(\Omega', \mathfrak{F}', \mu')$. By Alaoglu's theorem [1, V.4.2], W is $\tau_p(\Omega')$-compact. Define A as in Corollary 5; in this case, A is the intersection of all closed hyperplanes of measure 1. This implies that $\text{cor}_\varphi(\Omega) \subseteq A$.

Let $f \in X^*$, $\mu\{f = 0\} < 1$. There is $\epsilon > 0$ so that either $\mu\{f > \epsilon\} > 0$

or $\mu\{f < -\epsilon\} > 0$; assume without loss of generality that the first of these occurs. For $C = \{f \geq \epsilon\}$, if $x \in \mathrm{cor}_\varphi (C)$, then $f(x) \geq \epsilon$, so $A \cap \{f \neq 0\} \neq \emptyset$. So Corollary 5 is applicable. Thus the map $f \mapsto \int f \, d\mu'$ is $\tau_p(\Omega')$ - continuous, so $f \mapsto \int f \circ \varphi \, d\mu$ is weak* - continuous, and the Pettis integral $\int \varphi \, d\mu$ exists. The same argument shows that the Pettis integral $\int_C \varphi \, d\mu$ exists for any $C \in \mathfrak{F}$. □

Remarks. (a) In the terminology of [2], property (C) implies the PIP. (b) In the notation used above, the unit ball of A is W_1 , and $(W_1 , \tau_p(A))$ is the weak* topology there. This is homeomorphic to $(W_2 , \sigma(L^1, L^\infty))$, which is clearly an Eberlein compact. So the subspace A is isomorphic to a subspace of a WCG Banach space. However, A need not be separable.

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